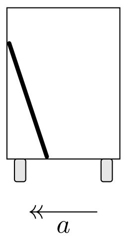
1801. Show that the area enclosed by the curve $y = x^2$ and the line $y = kx$ is given by $\left|\frac{1}{6}k^3\right|$.

1802. Simplify $\left(\sqrt{x} + \sqrt{x+1}\right)^3 + \left(\sqrt{x} - \sqrt{x+1}\right)^3$.

1803. A rigid, rectangular sheet of plywood of mass 10 kg is being moved in a van. The diagram shows the view forwards, inside the van, as it turns left. The sheet of plywood is at 75° to the vertical, and the floor is rough enough to ensure that it does not slide at any point. The wall of the van is modelled as smooth.



(a) Find the horizontal reaction force at the van wall before the turn, i.e. when $a = 0$.

(b) The van turns sharply. The top of the sheet of plywood leaves the wall, and the sheet stands vertically for a moment, at rest relative to the van. Show carefully that, if the van continues to turn with $a > 0$, then the sheet will fall towards the opposite wall.

1804. Disprove the following statement: "If u_n is a GP and f is a linear function, then $\text{f}(u_n)$ is also a GP."

1805. If $u = \sec x$, determine the exact value of $\left.\dfrac{du}{dx}\right|_{x=\frac{\pi}{6}}$

1806. Show that no (x, y) points satisfy both of

$$x^2 + (y+4)^2 < 4,$$
$$(x-2)^2 + y^2 < 6.$$

1807. Make y the subject of $x = \dfrac{y - \sqrt{y^2 + 16y}}{8}$.

1808. A student writes the following

$$\begin{aligned} &\ln x + \ln(x-1) = \ln 6 \\ \Longrightarrow\ & x + (x-1) = 6 \\ \Longrightarrow\ & x = \tfrac{7}{2}. \end{aligned}$$

Explain the error, and correct it.

1809. "The y axis is tangent to the curve $x = y^3 - y$." True or false?

1810. A sample x_i has mean $\bar{x}$ and standard deviation s_x. Explain why a coded sample $y_i = ax_i + b$ has mean $\bar{y} = a\bar{x} + b$ and variance $s_y^2 = a^2 s_x^2$.

1811. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $\sqrt[4]{x} = 2$,

② $x = 16$.

1812. Solve $\tan^2 2x - \sqrt{3}\tan 2x = 0$ for $x \in [0, \pi]$.

1813. Consider the curve $y = x^4 - x^3 - x^2 + x - 1$. You are given that $(1, -1)$ is a stationary point.

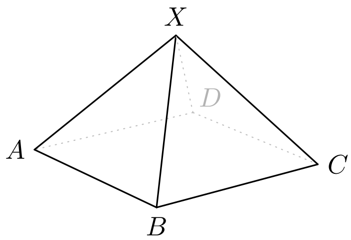
(a) Determine, without using a calculator, the x coordinates of all stationary points.

(b) Hence, give an accurate sketch of the curve. You don't need to calculate axis intercepts.

1814. From a committee of ten people, a chairperson and a secretary are to be chosen. Find the number of different ways in which this can be done.

1815. Prove that, for any constant $q \in \mathbb{R}$, $x^2 + qx + 1$ leaves a remainder when divided by $(qx - 1)$.

1816. The square-based pyramid shown below is formed of eight edges of unit length.



Determine the length of the shortest path from A to C, on the surface of the pyramid,

(a) if travel on the base is possible,

(b) if travel on the base is not possible.

1817. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $(x-1)e^{2x-3} = 0 \Longrightarrow x = 1$,

(b) $(x-1)\ln(2x-3) = 0 \Longrightarrow x = 1$.

1818. A function is given as $\text{f}(x) = \ln(4-x) - \ln x$.

(a) Find the largest real domain over which $\text{f}(x)$ can be defined.

(b) Show that $y = \text{f}(x)$ has a point of inflection on the x axis.

(c) By considering values close to the boundaries of the domain of $\text{f}(x)$, sketch $y = \text{f}(x)$.

1819. When chimneying, a rock-climber wedges him or herself between two parallel rock faces, back to one of them, feet against the other. In this question, take the faces to be vertical, the contact forces to act at the same horizontal level, and the climber's centre of mass to lie at a point dividing the width of the chimney in the ratio $1:4$.

(a) Explain why the reaction forces exerted on the climber by the two faces must have the same magnitude.

(b) Explain why, if the coefficients of friction are different at the two rock faces, the climber should face away from the rougher wall.

(c) If the walls have coefficients of friction $\frac{1}{3}$ and $\frac{1}{5}$, find the minimum reaction force required to maintain equilibrium.

1820. By considering the signs of the factors in a sketch, show that the inequality $(y-x^2)(y-4) \leq 0$ defines three distinct regions in the (x, y) plane.

1821. Describe all functions f for which f'' is quadratic.

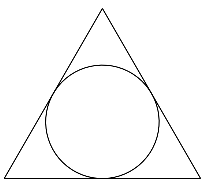
1822. Lines A_1 and A_2 have equations $y = \pm 2x \mp 6$. Point P is equidistant from A_1 and A_2. The locus of P consists of two lines B_1 and B_2. Find the equations of B_1 and B_2.

1823. For small θ, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

(a) Use $\theta = \frac{\pi}{6}$ to show that $\pi \approx 6\sqrt{2-\sqrt{3}}$.

(b) Calculate the percentage error, and comment.

1824. In the diagram below, an equilateral triangle has been circumscribed around a circle.



Prove that the ratio of areas is $3\sqrt{3} : \pi$.

1825. A game involves two dice: one is a regular cube, the other a regular dodecahedron. Both dice are numbered $1, 2, \ldots$. The dice are rolled together, and the total S is noted.

(a) Show that $\mathbb{P}(S = 3) = \frac{1}{36}$.

(b) Over a period of time, there are 10 instances when $S = 3$. Determine the expected number of these in which the dodecahedron showed 2.

1826. Show that no (x, y) pairs satisfy both of

$$x^2 - y^2 = 1,$$
$$16x^2 + y^2 < 15.$$

1827. In this question, the function p is a polynomial. State, with a reason, whether the curve $y = \mathrm{p}(x)$ necessarily intersects the following curves:

(a) $y = \mathrm{p}(-x)$,

(b) $y = -\,\mathrm{p}(x)$,

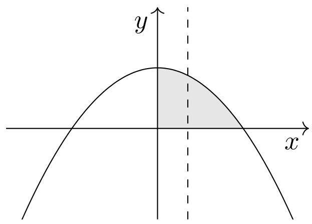
(c) $y = -\,\mathrm{p}(-x)$.

1828. In each case, find all possible values of the area of the given shape. Write your answers in interval set notation.

(a) parallelogram with side lengths (a, b, a, b),

(b) kite with side lengths (a, a, b, b).

1829. The axes below show the parabola $y = 1 - x^2$, and a line $x = k$, for some $k \in (0, 1)$.



(a) Show that, if the dashed line is to divide the shaded area in half, k must satisfy the equation $k^3 - 3k + 1 = 0$.

(b) Solve this equation using fixed-point iteration.

1830. A coin is tossed 6 times. Find the probability that no two consecutive tosses yield the same result.

1831. Two particles, projected at the same time from points on horizontal ground, have trajectories with the following Cartesian equations:

$$y = 8x - 2x^2,$$
$$2y = -x^2 + 14x - 33.$$

(a) By calculating the vertices of the parabolae, or otherwise, show that the particles are always at the same height as each other.

(b) The parabolae intersect at $(11/3, 22/9)$. Explain whether it is guaranteed that the particles will collide at this point.

1832. Write down $\displaystyle\int e^{2x+1}\,dx$.

1833. The curve $y = x^2 - x$ has two tangents that pass through the point $(6, 14)$.

(a) Show that the tangent at a general point $x = p$ has equation $y = (2p-1)x - p^2$.

(b) Hence, determine the equation of the tangents that pass through $(6, 14)$.

1834. Express $4z^2 + 10z + 19$ in terms of $(2z + 1)$.

1835. Show that the curves $y = \ln ax$ and $y = \ln bx$, for constants $a, b > 0$, are translations of one another, and give the translation vector.

1836. Simplify $\frac{d}{dx}(1 + y)^2 + \frac{d}{dx}(1 - y)^2$.

1837. In this question, S is the sum of the integers from 1 to 100 which are not multiples of 4.

(a) Explain why the sum of the integers from 1 to 100 which **are** multiples of 4 is given by

$$4 \times \tfrac{1}{2}n(n + 1)\Big|_{n=25}$$

(b) Hence, find S.

1838. Two dice have been rolled. Determine whether the fact "The individual scores differ by two" increases, decreases or doesn't change the probability that the combined score is ten.

1839. Solve $\log_{e^3} x + \ln x = 4$.

1840. For a general quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, the quadratic formula is given by

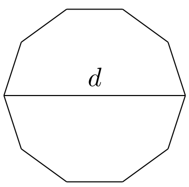
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In terms of a, b, c, find simplified expressions for

(a) the sum of the roots of a quadratic,

(b) the product of the roots of a quadratic.

1841. The diagram shows a regular $2n$-gon of side length l, with a diameter drawn from vertex to vertex.



Prove that $d = l \operatorname{cosec} \frac{90°}{n}$.

1842. Two sets of bivariate data have $r_1 = 0.386$ and $r_2 = 0.417$. In two individual hypothesis tests for correlation, each of these lies in the acceptance region. State, with a reason, whether a test on the combined set of data would necessarily yield a value for r lying in the new acceptance region.

1843. Prove that the quadratic $(3x^2 + x + 3)$ is not a factor of $27x^4 + 5x^3 - x - 9$.

1844. A differential equation is given as

$$\frac{dx}{dt} + \frac{x}{t} = t^2.$$

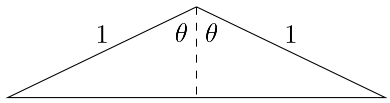
A quadratic solution $x = at^2 + bt + c$ is suggested.

(a) Find $\frac{dx}{dt}$ for the proposed solution.

(b) Hence, show that no such solution exists.

1845. Prove that, if $\mathbb{P}(A \mid B) > \mathbb{P}(B \mid A)$ for events A and B, then $\mathbb{P}(A) > \mathbb{P}(B)$.

1846. By finding the area of the isosceles triangle shown below in two different ways, prove the double-angle formula $\sin 2\theta \equiv 2 \sin\theta \cos\theta$.



1847. A *Reuleaux triangle* is a shape constructed from an equilateral triangle of side length r, by drawing an arc of radius r centred on each vertex.

(a) Sketch a Reuleaux triangle.

(b) Show that, in every orientation, a Reuleaux triangle has the same width r.

(c) Show that the area is $\frac{1}{2}(\pi - \sqrt{3})r^2$.

1848. Two sets I and J are defined as subsets of $\mathbb{R}$ by the values that satisfy the inequalities $x^2 - 2x - 24 < 0$ and $5x - 4 \geq 1$ respectively. Find the set $I \cap J$, giving your answer in interval set notation.

1849. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $(x - a)(x - b)(x - c) = 0$,

② $x \in \{a, b\}$.

1850. Describe the single transformation which takes the graph $y = 2e^x$ onto the graph $y = \ln \frac{1}{2}x$.

1851. A trapezium, which is not a parallelogram, has three of its sides described, in tip-to-tail order around the perimeter, by the vectors $\mathbf{a}$, $\mathbf{b}$ and $k\mathbf{a}$. Give the set of possible values of the scalar k.

1852. If $p = a^x$ and $q = b^x$, show that $p = q^{\log_b a}$.

1853. Show that the following implication holds:

$$12x = 3t^2 + \frac{1}{t^2} - 4t + 6$$
$$\Longrightarrow t\frac{dx}{dt} + 2x = t^2 - t + 1.$$

1854. Prove that there is only one way of shading five squares of a three-by-three grid such that no two shaded squares share a border.

1855. In this question, the continuous random variable X has a generic normal distribution: $X \sim N(\mu, \sigma^2)$. Probabilities are being numerically approximated using the trapezium rule.

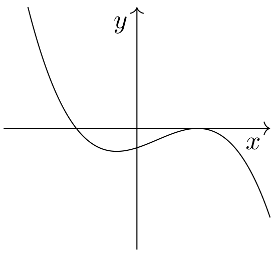
(a) Explain whether this approximating procedure will over or underestimate probabilities, for values of X

i. close to the mean,

ii. in the tails of the distribution.

(b) Explain why the trapezium rule will give good approximations at around $|X - \mu| = \sigma$.

1856. Determine whether $y = (1 - x)^2(1 + x)$ could be the equation generating the following graph:



1857. Simultaneous equations are given as

$$2\sqrt{x} + 2\sqrt{y} = 3,$$
$$y = x^2.$$

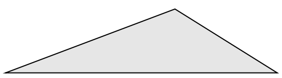
Solve the equations in simplified exact form.

1858. Write down the area scale factor when $y = \mathrm{f}(x)$ is transformed to $y = a\,\mathrm{f}(bx + c) + d$.

1859. Using the compound-angle formulae, express the following separable differential equation in the form $\mathrm{f}(y)\frac{dy}{dx} = \mathrm{g}(x)$ for some functions f and g:

$$\frac{dy}{dx} = \cos(x - y) - \cos(x + y).$$

1860. A triangle has side lengths in GP, with shortest side l and common ratio $3/2$.



Show that the area of the triangle is given by

$$A_\triangle = \frac{\sqrt{1463}}{64} l^2.$$

1861. For $X \sim N(0, 1)$, find $\mathbb{P}(X^2 - 1 > X)$.

1862. Write the following in terms of 4^x:

(a) 2^{6+2x},

(b) 8^{4-2x}.

1863. An operation $\star$, acting on two real numbers a, b, is defined as $a \star b = a^2 + ab + b^2$.

(a) Evaluate $\sqrt{3} \star \sqrt{27}$.

(b) Solve the equation $x \star (x + 2) = 1$.

1864. An iterative sequence is defined by

$$x_{n+1} = \frac{1}{x_n + 1}, \quad x_0 = a + b\sqrt{2}.$$

(a) Show that $x_2 = \dfrac{a + 1 + b\sqrt{2}}{a + 2 + b\sqrt{2}}$.

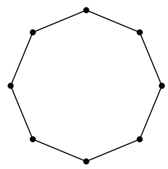
(b) Find an ordinal formula for x_n in terms of a, b and n, of the form $x_n = 1 - \mathrm{f}(n)$.

1865. State, with a reason, whether the following hold, with regard to a hypothesis test:

(a) "Some samples fall in neither the critical or acceptance regions."

(b) "If a sample has a test statistic which falls in the critical region, then its p-value is less than the significance level."

1866. Three distinct vertices are selected at random on a regular octagon.



Determine the probability that all three vertices are adjacent.

1867. The iteration $x_{n+1} = x_n^2 + px_n + q$ has exactly one fixed point. Find p in terms of q.

1868. In music theory, the intervals known as an *octave* and a *fifth* correspond to frequency ratios of 2 : 1 and 3 : 2 respectively. Prove that no interval, i.e. frequency ratio can be expressed both as a whole number of octaves and as a whole number of fifths.

1869. Find the area enclosed by the parabolae $y = x^2$ and $y = 1 + x - x^2$.

1870. Prove the following identities:

(a) $\sin(90° - \theta) \equiv \cos(\theta)$,

(b) $\sin(45° + \theta) \equiv \cos(45° - \theta)$.

1871. By proposing $p + q\sqrt{2}$ for $p, q \in \mathbb{N}$, and setting up simultaneous equations, determine the positive square root of $33 + 8\sqrt{2}$.

1872. Variables x and y have constant rates of change

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b.$$

Find the rate of change of xy in terms of x, y, a, b.

1873. Show that, although the second derivative of the curve $y = \sqrt[3]{x}$ is undefined at the origin, the curve is nevertheless inflected there.

1874. The equations $\mathrm{f}(x) = 0$ and $\mathrm{g}(x) = 0$, where f and g are distinct quadratic functions, have the same solution set S. The equation $\mathrm{f}(x) = \mathrm{g}(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",

(b) "S is a subset of the solution set of E,

(c) "the solution set of E is a subset of S".

Note that the empty set is a subset of every set.

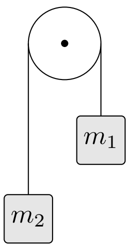
1875. A sequence is given, for constant $k > 0$, by

$$\ln k, \ln k^2, \ln k^3, ..., \ln k^9.$$

(a) Show that the sequence is arithmetic.

(b) Find the mean and median of the sequence, giving your answer in terms of k.

1876. Two masses $m_1 > m_2$ are attached to opposite ends of a light, inextensible string, which is passed over a fixed pulley.



The masses accelerate, but friction in the pulley is non-negligible, generating different tensions T_1 on the right and T_2 on the left of the pulley. Give, with justification, the order of the magnitudes of the four quantities m_1g, m_2g, T_1, T_2. Write your answer in the form $Q_a < Q_b < Q_c < Q_d$.

1877. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto \dfrac{1}{x+1}$,

(b) $x \mapsto \dfrac{1}{x^2+1}$,

(c) $x \mapsto \dfrac{1}{x^3+1}$.

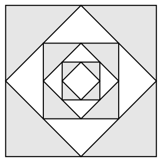
1878. Show that the following algebraic fraction cannot be simplified to a polynomial:

$$\frac{36x^3 + 216x^2 + 233x - 48}{3x + 5}.$$

1879. Find the 10^{th} and 90^{th} percentiles of the normal distribution $X \sim N(40, 9)$.

1880. Show that $\displaystyle\int_1^2 \sum_{i=1}^{3} x^{-i}\, dx = \frac{7}{8} + \ln 2$.

1881. A symmetrical fractal pattern uses rotated squares inscribed in squares, as shown.



By evaluating an infinite geometric sum, find the limit of the ratio of shaded to unshaded area.

1882. From the formulae $s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$, show algebraically that $s = \frac{1}{2}(u \pm v)t$.

1883. The curve $y = x^6 + x$ has a second derivative which is zero at the origin. Show that this is not a point of inflection.

1884. Find simplified expressions for the sets

(a) $(A \setminus B) \cap A$,

(b) $(A \cap B) \setminus B$,

(c) $(A \setminus B) \cup B$.

1885. A set of two linear equations in three unknowns is given as follows:

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0. \end{aligned}$$

Is the following statement true or false?

"There is no set $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ of constants for which these equations yield a unique point (x, y, z) as a solution."

1886. Two workmen of equal height are carrying a long plank on their shoulders. The plank is 4 metres long, has a mass of 10 kg, and is uniform and rigid. The men are 0.5 m and 1 m from their respective ends. Find the magnitude of the total contact force each exerts on the plank when

(a) they are at rest on horizontal ground,

(b) they are moving at constant speed up a steady slope of 10°.

1887. By unwrapping its curved surface to form a sector, prove that the total surface area of a right-circular cone is given, in terms of the radius r and slant height l, by $A = \pi r(r + l)$.

1888. The function g has domain $[0, 1]$ and range $[0, 1]$. State, with a reason, whether the following are well-defined functions over $[0, 1]$:

(a) $x \mapsto \mathrm{g}\big(1 + \mathrm{g}(x)\big)$,

(b) $x \mapsto \mathrm{g}\big(1 - \mathrm{g}(x)\big)$.

1889. An arithmetic series has first term 15, common difference 7, and partial sum $S_n = 29385$. Find n.

1890. State the possible numbers of intersections of the graphs $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$, when they are

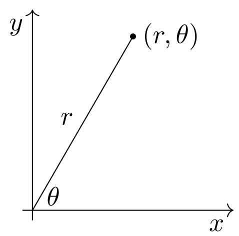
(a) a cubic and a quadratic,

(b) two cubics,

(c) a cubic and a quartic.

1891. Samples are taken from a large population, whose distribution is modelled as approximately normal. The standard deviation of the means of samples of size k is 0.12. Give the standard deviation of the means of samples of size $4k$.

1892. *Polar coordinates* (r, θ) are defined as the distance from O and the angle at subtended at O, measured in the usual sense anticlockwise from the positive x axis.



(a) Express x and y in terms of r and θ.

(b) Express $\tan\theta$ and $\cot\theta$ in terms of x and y.

(c) Hence, by finding the Cartesian equation, show that the curve $r\sin^2\theta = \cos\theta$ is a parabola.

1893. A function has second derivative $\mathrm{f}''(x) = 4x$, and $\mathrm{f}'(2) = 6$. Show that the function is stationary at $x = \pm\sqrt{2}$.

1894. Show that the solution set of $3\sin^2 x - \cos^2 x = 0$ is $\{x : x = 180°n \pm 30°, n \in \mathbb{Z}\}$.

1895. In terms of d and g, find the minimum speed at which it is possible to throw a projectile to a friend who is d metres away horizontally.

1896. Show that it is impossible to find constants P, Q such that the following is an identity:

$$\frac{1}{x^3 + x} \equiv \frac{P}{x} + \frac{Q}{x^2 + 1}.$$

1897. The cubic approximation for $\sin\theta$, valid for small angles defined in radian measure, is $\sin\theta \approx \theta - \frac{1}{6}\theta^3$. Show that the area of a segment, subtending a small angle θ at the centre of a circle of radius r, may be approximated as $A \approx \frac{1}{12}r^2\theta^3$.

1898. Assuming $\dfrac{d}{dx}e^x = e^x$, prove that $\dfrac{d}{dx}2^x = \ln 2 \cdot 2^x$.

1899. A sample $\{x_i\}$ is taken, and the sample variance s_x^2 is calculated. Afterwards, 20% of x_i values, chosen at random, are increased by a quarter. Find the expected percentage change in s_x^2.

1900. Solve $\dfrac{2\ln x}{\ln(5x - 4)} = 1$.

—— End of 19th Hundred ——